On one dimensional Leibniz central extensions of a naturally graded filiform Lie algebra

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Abstract

   This paper deals with the classification of Leibniz central extensions of a naturally graded filiform Lie algebra. We choose a basis with respect to that the table of multiplication has a simple form. In low dimensional cases isomorphism classes of the central extensions are given. In parametric family orbits cases invariant functions (orbit functions) are provided.

1   Introduction

   Leibniz algebras were introduced by J.-L.Loday [12],[14]. (For this reason, they have also been called "Loday algebras"). A skew-symmetric Leibniz algebra is a Lie algebra. The Leibniz algebras play an important role in Hochschild homology theory, as well as in Nambu mechanics. The main motivation of J.-L.Loday to introduce this class of algebras was the search of an "obstruction" to the periodicity of algebraic $K-$theory. Beside this purely algebraic motivation some relationships with classical geometry, non-commutative geometry and physics have been recently discovered. Leibniz algebras appear to be related in a natural way to several topics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, quantum physics etc., as a generalization of the corresponding applications of Lie algebras to these topics. It is a generalization of Lie algebras. K.A. Umlauf (1891) initiated the study of the simplest non-trivial class of Lie algebras. In his thesis he presented the list of Lie algebras of dimension less than ten admitting a so-called adapted basis (now, Lie algebras with this property are called filiform Lie algebras).There is a description of naturally graded complex filiform Lie algebras as follows: up to isomorphism there is only one naturally graded filiform Lie algebra in odd dimensions and they are two in even dimensions. With respect to the adapted basis table of multiplications have a simple form. Since a Lie algebra is Leibniz it has a sense to consider a Leibniz central extensions of the filiform Lie algebra. The resulting algebra is a filiform Leibniz algebra and it is in of interest to classify these central extensions. In the present paper we propose an approach based on algebraic invariants. The results show that this approach is quite effective in the classification problem. As a final result we give a complete list of the mentioned class of algebras in low dimensions. In parametric family orbits case we provide invariant functions to discern the orbits (orbit functions). As the next step of the study the algebraic classification may be used in geometric study of the algebraic variety of filiform Leibniz algebras.

   The (co)homology theory, representations and related problems of Leibniz algebras were studied by Loday, J.-L. and Pirashvili, T. [14], Frabetti, A. [6] and others. A good survey about these all and related problems is [13].

   The problems related to the group theoretical realizations of Leibniz algebras are studied by Kinyon, M.K., Weinstein, A. [10] and others.

   Deformation theory of Leibniz algebras and related physical applications of it, is initiated by Fialowski, A., Mandal, A., Mukherjee, G. [5].

   The outline of the paper is as follows. Section 2 is a gentle introduction to a subclass of Leibniz algebras that we are going to investigate. Section 3 describes the behavior of parameters under the isomorphism action (adapted changing). Sections 3.1 — 3.5 contain the main results of the paper consisting of the complete classification of one dimensional Leibniz central extensions of low dimensional graded filiform Lie algebras. Here we give complete lists of all one dimensional Leibniz central extensions in low

dimensions cases. We distinguish the isomorphism classes and show that they exhaust all possible cases. For parametric family cases the corresponding invariant functions are presented. Since the proofs from the considered cases can be carried over to the other cases by minor changing we have chosen to omit the proofs of them. All details of the other proofs are available from the authors.

2   Preliminaries

Let V be a vector space of dimension n over an algebraically closed field K (char$K$=0). The bilinear maps $V \times V \longrightarrow V$ form a vector space $Hom(V \otimes V, V)$ of dimension n^3, which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $Alg_n(K) \cong K^{n^3}$. An n-dimensional algebra L over K may be considered as an element of $Alg_n(K)$ via the bilinear mapping $[\cdot, \cdot] : L \times L \longrightarrow L$ defining a binary algebraic operation on L. Let $\{e_1, e_2, ..., e_n\}$ be a basis of L. Then the table of multiplication of L is represented by a point $\{\gamma_{ij}^k\}$ of the affine space K^{n^3} as follows:

$$[e_i, e_j] = \sum_{k=1}^{n} \gamma_{ij}^k e_k$$

(γ_{ij}^k are called structural constants of L). The linear group $GL_n(K)$ acts on $A\lg_n(K)$ by

$$[x, y]_{g*L} = g[g^{-1}(x), g^{-1}(y)]_L$$

where $L \in A\lg_n(K), g \in GL_n(K)$

Two algebras L_1 and L_2 are isomorphic if and only if they belong to the same orbit under this action.

Definition 2.1. *An algebra L over a field K is said to Leibniz algebra if its bilinear operation $[\cdot, \cdot]$ satisfies the following Leibniz identity:*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

Let $LB_n(K)$ be the subvariety of $Alg_n(K)$ consisting of all n-dimensional Leibniz algebras over K. It is invariant under the above mentioned action of $GL_n(K)$. As a subset of $Alg_n(K)$ the set $LB_n(K)$ is specified by the system of equations with respect to the structural constants γ_{ij}^k:

$$\sum_{l=1}^{n} (\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m) = 0$$

Further all algebras are assumed to be over the field of complex numbers $\mathbb{C}$.

Definition 2.2. *Let L and V be Leibniz algebras. An extension $\widetilde{L}$ of L by V is a short exact sequence:*

$$0 \longrightarrow V \longrightarrow \widetilde{L} \longrightarrow L \longrightarrow 0$$

of Leibniz algebras.

The extension is said to be *central* if the image of V is contained in the center of $\widetilde{L}$ and *one dimensional* if V is.

Let L be a Leibniz algebra. We put:

$$L^1 = L, \ L^{k+1} = [L^k, L], \ k \geq 1.$$

Definition 2.3. *A Leibniz algebra L is said to be nilpotent if there exists an integer $s \in N$, such that*

$$L^1 \supset L^1 \supset ... \supset L^s = \{0\}.$$

Definition 2.4. *A Leibniz algebra L is said to be filiform if $dim L^i = n - i$, where $n = dim L$ and $2 \leq i \leq n$.*

It is obvious that a filiform Leibniz algebra is nilpotent.

The set of all $n-$dimensional filiform Leibniz algebras we denote as $Leib_n$.

3   Simplifications in CE_{μ_n}

In this section we consider a subclass of $Leib_{n+1}$ called truncated filiform Leibniz algebras in [?], where motivations to study of this case also has been given. According to [?] the table of multiplication of the truncated filiform Leibniz algebras can be represented as follows:

$$
\begin{cases}
[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n - 1, \\
[e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n - 1, \\
[e_0, e_0] = b_{0,0} e_n, \\
[e_0, e_1] = -e_2 + b_{01} e_n, \\
[e_1, e_1] = b_{11} e_n, \\
[e_i, e_j] = b_{ij} e_n, & 1 \leq i < j \leq n - 1, \\
[e_i, e_j] = -[e_j, e_i], & 1 \leq i < j \leq n - 1, \\
[e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b \, e_n & 1 \leq i \leq n - 1.
\end{cases}
$$

$b \in \{0, 1\}$ for odd n and $b = 0$ for even n,

The basis $\{e_0, e_1, ..., e_{n-1}, e_n\}$ leading to this representation is said to be *adapted*.

It is obvious that this is a class of all one dimensional Leibniz central extensions of the graded filiform Lie algebra with the composition law $[\cdot, \cdot]$:

$$
\mu_n : \quad [e_i, e_0] = e_{i+1}, \quad 1 \leq i \leq n - 1,
$$

with respect to the adapted basis $\{e_0, e_1, ..., e_{n-1}\}$.

Definition 3.1. *Let $\{e_0, e_1, ..., e_n\}$ be an adapted basis of $L \in CE(\mu_n)$. Then a nonsingular linear transformation $f : L \to L$ is said to be adapted if the basis $\{f(e_0), f(e_1), ..., f(e_n)\}$ is adapted.*

The set of all adapted elements of GL_{n+1} is a subgroup and it is denoted by G_{ad}.

Elements of $CE(\mu_n)$ represented by the above table shortly we denote as $L = L(b_{0,0}, b_{0,1}, b_{1,1}, ..., b_{i,j})$ with $1 \leq i < j \leq n - 1$.

Since a filiform Leibniz algebra is 2-generated the basis changing on it can be taken as follows:

$$
f(e_0) = \sum_{i=0}^{n} A_i e_i
$$

$$
f(e_1) = \sum_{i=0}^{n} B_i e_i
$$

where $A_0(A_0 B_1 - A_1 B_0)(A_0 + A_1 b) \neq 0$ and let $f(L) = L'$.

The following lemma specifies the parameters $(b_{00}, b_{01}, b_{11}, ..., b_{ij})$ of the algebra $L = L(b_{0,0}, b_{0,1}, b_{1,1}, ..., b_{i,j})$.

Lemma 3.1. *Let $L \in CE(\mu_n)$. Then the following equalities hold:*

1.

$$
b_{i+1,j} = -b_{i,j+1} \quad 1 \leq i, j \leq n - 1, \quad i + j \neq n
$$

2.

$$
b_{1,2i+1} = 0 \qquad 0 < i \leq \left[\frac{n-2}{2}\right]
$$

Proof. 1. From Leibniz Identity we will get the following identity for $i, j \geq 2$

$$
\begin{aligned}
b_{i+1,j}\, e_n + b_{i,j+1}\, e_n &= [e_{i+1}, e_j] + [e_i, e_{j+1}] \\
[[e_i, e_0], e_j] + [e_i, [e_j, e_0]] &= [[e_0, e_j], e_i] - [[e_0, e_i], e_j] = [e_0, [e_j, e_i]] = 0 \Rightarrow b_{i+1,j} = -b_{i,j+1}.
\end{aligned}
$$

The equality still true for $i, j \geq 1$

    *2.*This equality is followed from the chain of equalities

$$
\begin{aligned}
b_{1,2i+1}\, e_n &= [e_1, e_{2i+1}] = [e_1, [e_{2i}, e_0]] \\
&= [[e_1, e_{2i}], e_0] - [[e_1, e_0], e_{2i}] \\
&= [[e_1, e_0], e_{2i}] = [e_2, e_{2i}] = 0
\end{aligned}
$$

$\square$

Consequence of this lemma we will get $b_{i+2,i} = b_{i,i+2} = 0$ and

$$
b_{i,j} = -b_{i-1,j+1} = b_{i-2,j+2} = \ldots = (-1)^i\, b_{i-(i-1),j+(i-1)} = (-1)^i\, b_{1,j+i-1}.
$$

Proposition 3.1. *Let $f \in G_{ad}$ if $L \in CE(\mu_n)$ then f has the following form:*

$$
\begin{aligned}
e'_0 &= \sum_{i=0}^{n} A_i e_i \\
e'_i &= \sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_i + (*)e_n \qquad\qquad 1 \leq i \leq n-1 \\
e'_n &= A_0^{n-2} B_1 (A_0 + A_1 b)\, e_n
\end{aligned}
$$

where $A_0 B_1 (A_0 + A_1 b) \neq 0$

Proof. Note that

$$
e'_i = f(e_i) = [f(e_{i-1}), f(e_0)] = \sum_{j=i}^{n-1} A_0^{i-2}(A_0 B_{j-i+1} - A_{j-i+1} B_0)e_j + (*)e_n, \ 2 \leq i \leq n-1 \quad (1)
$$

$$
e'_n = f(e_n) = [f(e_{n-1}), f(e_0)] = A_0^{n-3}(A_0 B_1 - A_1 B_0)(A_0 + A_1 b)e_n \quad (2)
$$

$$
\text{Consider } [f(e_2), f(e_1)] = B_0 \sum_{i=3}^{n-1}(A_0 B_{i-2} - A_{i-2} B_0)e_i + (*)e_n,
$$

and equating the corresponding coefficients we get $B_0(A_0 B_{i-2} - A_{i-2} B_0) = 0$, $3 \leq i \leq n-1$ .Since $A_0 B_1 - A_1 B_0 \neq 0$ then this relation is only possible if $B_0 = 0$.

$\square$

Definition 3.2. *The following transformations of L is said to be elementary:*

$$
\sigma(b, k) = \begin{cases} f(e_0) = e_0, \\ f(e_1) = e_1 + b\, e_k, & b \in \mathbb{C}, \quad 2 \leq k \leq n \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1, \end{cases}
$$

$$
\tau(a, k) = \begin{cases} f(e_0) = e_0 + a\, e_k, & a \in \mathbb{C} \quad 1 \leq k \leq n, \\ f(e_1) = e_1, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1, \end{cases}
$$

$$
\upsilon(a, b) = \begin{cases} f(e_0) = a\, e_0, \\ f(e_1) = b\, e_1, & a, b \in \mathbb{C}^*. \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1 \end{cases}
$$

Proposition 3.2. *Let f be an adapted transformation of L. Then it can be represented as composition:*

$$f = \tau(a_n, n) \circ \tau(a_{n-1}, n-1) \circ ... \circ \tau(a_2, 2) \circ \sigma(b_n, n) \circ \sigma(b_{n-1}, n-1) \circ ... \circ \sigma(b_2, 2) \circ \tau(a_1, 1) \circ \upsilon(a_0, b_1),$$

Proof. The proof is straightforward.   □

Proposition 3.3. *The transformation*

$$g = \tau(a_n, n) \circ \tau(a_{n-1}, n-1) \circ ... \circ \tau(a_2, 2) \circ \sigma(b_n, n) \circ \sigma(b_{n-1}, n-1)$$

does not change the structural constants of this case

So from the assertion above of proposition 3.3. we have the adapted transformations are reduced to the transformation of the form :

$$\begin{cases} f(e_0) = A_0\, e_0 + A_1\, e_1 \\ f(e_1) = B_1\, e_1 + B_2\, e_2 + ... + B_{n-2}\, e_{n-2}, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1, \end{cases}$$

where $A_0 B_1 (A_0 + A_1 b) \neq 0$

Under the action of the given basis change we have

The next lemma defines the action of the adapted changing of basis to the structural constants of algebras from $CE(\mu_n)$.

Lemma 3.2. *Let $L \in CE(\mu_n)$ with parameters $L(\alpha)$ where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, ..., b_{1,2j})$ and L' be the image of L under the action of G_{ad}. Then for parameters of L' one has:*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-2} B_1 (A_0 + A_1 b)}$$

$$b'_{0,1} = \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^{n-2}(A_0 + A_1 b)},$$

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^{n-2}(A_0 + A_1 b)},$$

$$b'_{1,2j} = \frac{1}{B_1(A_0 + A_1 b)} \left(\sum_{k=1}^{n-1} \sum_{l=2j}^{n-k-1} (-1)^{k-1} A_0^{1+2j-n} B_k B_{l-2j+1} b_{1,k+l-1} + \sum_{k=1}^{n-2} (-1)^k A_0^{1+2j-n} B_k B_{n-k-2j+1} b \right),$$

where $l + k \neq n$.

Proof. Consider the product $[f(e_0), f(e_0)] = b'_{0,0} f(e_n)$. Equating the coefficients of e_n in it we get

$$A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1} = b'_{0,0} A_0^{n-2} B_1 (A_0 + A_1 b).$$

$$\text{Then } b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-2} B_1 (A_0 + A_1 b)}.$$

The product $[f(e_1), f(e_1)] = b'_{1,1} f(e_n)$ yields

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^{n-2}(A_0 + A_1 b)}.$$

Consider the equality

$$b'_{0,1} f(e_n) = [f(e_1), f(e_0)] + [f(e_0), f(e_1)].$$

Then $b'_{0,1}A_0^{n-2}B_1(A_0 + A_1b) = A_0B_1b_{0,1} + 2A_1B_1b_{1,1}$ and it implies that

$$b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^{n-2}(A_0 + A_1b)}.$$

According to Proposition 3.3.

$$
\begin{aligned}
e'_0 &= A_0\, e_0 + A_1\, e_1 \\
e'_1 &= B_1\, e_1 + B_2\, e_2 + \ldots + B_{n-2}\, e_{n-2} \\
e'_i &= \sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_i + (*)e_n \qquad\qquad 2 \le i \le n-1 \\
e'_n &= A_0^{n-2} B_1(A_0 + A_1b)\, e_n,
\end{aligned}
$$

then

$$
\begin{aligned}
[e'_i, e'_j] &= [\sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_k + (*)e_n, \sum_{l=j}^{n-1} A_0^{j-1} B_{l-j+1} e_l + (*)e_n], \\
&= [\sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_k, \sum_{l=j}^{n-1} A_0^{j-1} B_{l-j+1} e_l] \\
&= \sum_{k=i}^{n-1}\sum_{l=j}^{n-1} A_0^{i+j-2} B_{k-i+1} B_{l-j+1} [e_k, e_l] \\
&= \sum_{k=i}^{n-1}\sum_{l=j}^{n-k} A_0^{i+j-2} B_{k-i+1} B_{l-j+1} b_{k,l}\ \ e_n.
\end{aligned}
$$

Hence the equality

$$b'_{i,j}\, e'_n = [e'_i, e'_j]$$

gives the relation

$$b'_{i,j}A_0^{n-2}B_1(A_0 + A_1b) = \sum_{k=i}^{n-1}\sum_{l=j}^{n-k} A_0^{i+j-2} B_{k-i+1} B_{l-j+1} b_{k,l},$$

and then

$$b'_{i,j} = \frac{1}{B_1(A_0 + A_1b)}(\sum_{k=i}^{n-1}\sum_{l=j}^{n-k} A_0^{i+j-n} B_{k-i+1} B_{l-j+1} b_{k,l}).$$

from above **lemma 3.1.** if $b_{i,j} \neq 0$ can be representative as $b_{1,2j}$ so final formula will be :

$$b'_{1,2j} = \frac{1}{B_1(A_0 + A_1b)} \left(\sum_{k=1}^{n-1}\sum_{l=2j}^{n-k-1} (-1)^{k-1} A_0^{1+2j-n} B_k B_{l-2j+1} b_{1,k+l-1} + \sum_{k=1}^{n-2} (-1)^k A_0^{1+2j-n} B_k B_{n-k-2j+1} b \right)$$

where $l + k \neq n$.

$\square$

The next sections deal with the applications of the results of the previous section to the classification problem of $CE(\mu_n)$ at $n = 5 - 9$. It should be mentioned that the classifications of all complex nilpotent Leibniz algebras in dimensions at most 4 have been done before in [2].

Here to classify algebras from $CE(\mu_n)$ in each fixed dimensional case we represent it as a disjoin union of its subsets. Some of these subsets are single orbits and the others contain infinitely many orbits. In the last case we give invariant functions to discern the orbits.

To simplify calculation we will introduced the following notations:

$$\Delta = b_{0,1}^2 - 4\,b_{0,0}\,b_{1,1} \quad \text{and} \quad \Delta' = b_{0,1}'^2 - 4\,b_{0,0}'\,b_{1,1}'.$$

3.1   Central extension for 4-dimensional Lie algebra $CE(\mu_4)$

This section is devoted to the complete classification of $CE(\mu_4)$. According to our notations the elements of $CE(\mu_4)$ will be denoted by $L(\alpha)$, where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2})$. Note that in this case n is even then $b = 0$ (see the multiplication table of $CE(\mu_n)$.)

Theorem 3.1. *(Isomorphism criterion for $CE(\mu_4)$) Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from $CE(\mu_4)$ are isomorphic iff there exist $A_0, A_1, B_1 \in \mathbb{C}$: such that $A_0 B_1 \neq 0$ and the following equalities hold:*

$$b_{0,0}' \quad = \quad \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1}, \tag{3}$$

$$b_{1,1}' \quad = \quad \frac{B_1 b_{1,1}}{A_0^3}, \tag{4}$$

$$b_{0,1}' \quad = \quad \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^3}, \tag{5}$$

$$b_{1,2}' \quad = \quad \frac{B_1 b_{1,2}}{A_0^2}. \tag{6}$$

Proof. " If " part due to Lemma 3.2.

" Only if part."

Let the equations (3) – (6) hold. Then the following basis changing is adapted and it transforms $L(\alpha)$ to $L(\alpha')$

$$
\begin{aligned}
e_0' &= A_0 e_0 + A_1 e_1, \\
e_1' &= B_1 e_1, \\
e_2' &= A_0 B_1 e_2 + A_1 B_1 b_{1,1} e_4, \\
e_3' &= A_0{}^2 B_1 e_3 - A_1 A_0 B_1 b_{1,2} e_4, \\
e_4' &= A_0{}^3 B_1 e_4.
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
[e_0', e_0'] &= A_0{}^2 b_{0,0} e_4 + A_0 A_1 (-e_2 + b_{0,1} e_4) + A_1 A_0 e_2 + A_1{}^2 b_{1,1} e_4 \\
&= \frac{\left(A_0{}^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1{}^2 b_{1,1}\right)}{A_0^3 B_1} \; A_0^3 B_1 e_4 = b_{0,0}'\, e_4'.
\end{aligned}
$$

By the same steps we can get the second equation

$$
\begin{aligned}
[e_0', e_1'] &= -A_0 B_1 e_2 + A_0 B_1 b_{0,1} e_4 + A_1 B_1 b_{1,1} e_4 \\
&= -A_0 B_1 e_2 - A_1 B_1 b_{1,1} e_4 + A_0 B_1 b_{0,1} e_4 + 2 A_1 B_1 b_{1,1} e_4 \\
&= -e_2' + B_1 \left(A_0 b_{0,1} + 2 A_1 b_{1,1}\right) e_4 \\
&= -e_2' + b_{0,1}' A_0^3 B_1 e_4 = -e_2' + b_{0,1}' e_4
\end{aligned}
$$

$$[e_1', e_1'] = B_1{}^2 b_{1,1} e_4$$
$$= A_0^3 B_1 b_{1,1}' e_4 = b_{1,1}' \, e_4$$

$$[e_1', e_2'] = B_1{}^2 A_0 b_{1,2} e_4$$
$$= A_0^3 B_1 b_{1,2}' e_4 = b_{1,2}' \, e_4$$

$\square$

In this section we give a list of all algebras from $CE(\mu_4)$.

Represent $CE(\mu_4)$ as a union of the following subsets :

$U_1 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} \neq 0, b_{1,2} \neq 0\}$

$U_2 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} \neq 0, b_{1,2} = 0, \Delta \neq 0\}$

$U_3 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} \neq 0, b_{1,2} = \Delta = 0\}$

$U_4 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} \neq 0\}$

$U_5 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} = 0\}$

$U_6 = \{L\alpha) \in CE(\mu_4) \ : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} \neq 0\}$

$U_7 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} = 0\}$

$U_8 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} = b_{0,1} = b_{0,0} = 0, b_{1,2} \neq 0\}$

$U_9 = \{L(\alpha) \in CE(\mu_4) \ : b_{1,1} = b_{0,1} = b_{0,0} = b_{1,2} = 0\}$

Proposition 3.4.

1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \left(\frac{b_{1,2}'}{b_{1,1}'}\right)^4 \Delta'$$

2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_1$:

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \lambda$$

.

Then algebras from the set U_1 can be parameterized as $L(\lambda, 0, 1, 1)$, $\lambda \in C$.

Proof. $\Rightarrow$

Let $L(\alpha)$ and $L(\alpha')$ be isomorphic. Then due to theorem 3.1 there are a complex numbers

A_0, A_1 and B_1 : $A_0 B_1 \neq 0$ such that the action of the adapted group G_{ad} can be expressed by the following system of equations

$$b_{0,0}' = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1}, \tag{7}$$

$$b_{1,1}' = \frac{B_1 b_{1,1}}{A_0^3}, \tag{8}$$

$$b_{0,1}' = \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^3}, \tag{9}$$

$$b_{1,2}' = \frac{B_1}{A_0^2} b_{1,2}. \tag{10}$$

Then the one can easy to say that:

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta'$$

$\Leftarrow$

Let suppose the equality

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta'$$

holds

Consider the basis changing

$$e'_0 = \sum_{i=0}^{4} A_i e_i$$

$$e'_i = \sum_{k=i}^{3} A_0^{i-1} B_{k-i+1} e_i + (*)e_4 \qquad\qquad 1 \le i \le 3$$

Where $A_0 = \frac{b_{1,1}}{b_{1,2}}$, $A_1 = -\frac{b_{0,1}}{2\,b_{1,2}}$, and $B_1 = \frac{b_{1,1}{}^2}{b_{1,2}{}^3}$.This changing leads $L(\alpha)$to $L\left(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta, 0, 1, 1\right)$
The basis changing

$$e''_0 = \sum_{i=0}^{4} A'_i e'_i$$

$$e''_i = \sum_{k=i}^{3} A_0'^{i-1} B'_{k-i+1} e'_i + (*)e'_4 \qquad\qquad 1 \le i \le 3$$

Where $A'_0 = \frac{b'_{1,1}}{b'_{1,2}}$, $A'_1 = -\frac{b'_{0,1}}{2\,b'_{1,2}}$, and $B'_1 = \frac{b'_{1,1}{}^2}{b'_{1,2}{}^3}$. This changing leads $L(\alpha')$ to $L\left(\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta', 0, 1, 1\right)$
but by the hypothesis of the theorem

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta'$$

so $L(\alpha)$ and $L(\alpha')$ are isomorphic to the same algebra and therefore they are isomorphic. $\square$

Proposition 3.5.

1.   *Algebras from U_2 are isomorphic to L(1,0,1,0);*

2.   *Algebras from U_3 are isomorphic to L(0,0,1,0);*

3.   *Algebras from U_4 are isomorphic to L(0,1,0,1);*

4.   *Algebras from U_5 are isomorphic to L(0,1,0,0);*

5.   *Algebras from U_6 are isomorphic to L(1,0,0,1);*

6.   *Algebras from U_7 are isomorphic to L(1,0,0,0);*

7.   *Algebras from U_8 are isomorphic to L(0,0,0,1);*

8.   *Algebras from U_9 are isomorphic to L(0,0,0,0).*

Proof.

We will show that $U_2, ..., U_9$ are single orbit .To show for each subsets we find the corresponding basis changing leading to indicated in representative

For U_2

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2 + A_1 B_1 b_{1,1} e_4, \\
e_3' &= A_0{}^2 B_1 e_3, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0 = \frac{\Delta^{\frac{1}{4}}}{\sqrt{2}}, A_1 = \frac{-b_{0,1} \Delta^{\frac{1}{4}}}{2\sqrt{2}\, b_{1,1}} \text{and } B_1 = \frac{\Delta^{\frac{3}{4}}}{2\sqrt{2}\, b_{1,1}}
$$

For U_3

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2 + A_1 B_1 b_{1,1} e_4, \\
e_3' &= A_0{}^2 B_1 e_3, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0 \in \mathbb{C}^* \quad , \quad A_1 = \frac{-A_0 b_{0,1}}{2b_{1,1}} \text{and } B_1 = \frac{A_0{}^3}{b_{1,1}}
$$

For U_4

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2, \\
e_3' &= A_0{}^2 B_1 e_3 - A_0 A_1 B_1 b_{1,2} e_4, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0 = \sqrt{b_{0,1}}, A_1 = -\frac{b_{0,0}}{\sqrt{b_{0,1}}} \text{ and } B_1 = \frac{b_{0,1}}{b_{1,2}}
$$

For U_5

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2, \\
e_3' &= A_0{}^2 B_1 e_3, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0 = \sqrt{b_{0,1}}, A_1 = -\frac{b_{0,0}}{\sqrt{b_{0,1}}} \text{ and } B_1 \in \mathbb{C}^*
$$

For U_6

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2, \\
e_3' &= A_0{}^2 B_1 e_3 - A_0 A_1 B_1 b_{1,2} e_4, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0 = \sqrt[3]{b_{0,0} b_{1,2}}, \quad A_1 \in \mathbb{C} \text{ and } B_1 = \frac{b_{0,0}}{\sqrt[3]{b_{0,0} b_{1,2}}}
$$

For U_7

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2, \\
e_3' &= A_0{}^2 B_1 e_3, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0 \in \mathbb{C}^* \quad , \quad A_1 \in \mathbb{C} \ \text{ and } B_1 = \frac{b_{0,0}}{A_0}
$$

For U_8

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2, \\
e_3' &= A_0{}^2 B_1 e_3 - A_0 A_1 B_1 b_{1,2} e_4, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0 \in \mathbb{C}^* \quad , \quad A_1 \in \mathbb{C} \ \text{ and } B_1 = \frac{A_0{}^2}{b_{1,2}}
$$

For U_9

$$
\begin{aligned}
e_2' &= A_0 B_1 e_2, \\
e_3' &= A_0{}^2 B_1 e_3, \\
e_4' &= A_0{}^3 B_1 e_4
\end{aligned}
$$

where

$$
A_0, B_1 \in \mathbb{C}^* \quad , \quad A_1 \in \mathbb{C} \ \text{ and }
$$

$\square$

3.2   Central extension for 5-dimensional Lie algebra $CE(\mu_5)$

From Leibniz Identity we can show that $b = b_{2,3} = b_{1,4}$
Further the elements of $CE(\mu_5)$ will be denoted by $L(\alpha)$, where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b)$ meaning that they are depending on parameters $b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b$.

Theorem 3.2. *(Isomorphism criterion for $CE(\mu_5)$) Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from $CE(\mu_5)$ are isomorphic iff $\exists\, A_0, A_1, B_1 \in \mathbb{C}$: such that $A_0 B_1 (A_0 + A_1 b) \neq 0$,and the following equalities hold:*

$$
b_{0,0}' = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1 (A_0 + A_1 b)}, \tag{11}
$$

$$
b_{0,1}' = \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^3 (A_0 + A_1 b)}, \tag{12}
$$

$$
b_{1,1}' = \frac{B_1 b_{1,1}}{A_0^3 (A_0 + A_1 b)}, \tag{13}
$$

$$
b_{1,2}' = \frac{B_1{}^2 b_{1,2} + (-2 B_1 B_3 + B_2^2) b}{A_0{}^2 B_1 (A_0 + A_1 b)}, \tag{14}
$$

$$
b' = \frac{B_1 b}{(A_0 + A_1 b)}. \tag{15}
$$

Proof.

$\square$

In this section we give a list of all algebras from $CE(\mu_5)$.

Represent $CE(\mu_5)$ as a union of the following subsets:

$U_1 = \{L(\alpha) \in CE(\mu_5) \; : b \neq 0, b_{1,1} \neq 0\}$

$U_2 = \{L(\alpha) \in CE(\mu_5) \; : b \neq 0, b_{1,1} = 0, b_{0,1} \neq 0\}$

$U_3 = \{L(\alpha) \in CE(\mu_5) \; : b \neq 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0\}$

$U_4 = \{L(\alpha) \in CE(\mu_5) \; : b \neq 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\}$

$U_5 = \{L(\alpha) \in CE(\mu_5) \; : b = 0, b_{1,1} \neq 0, b_{1,2} \neq 0\}$

$U_6 = \{L(\alpha) \in CE(\mu_5) \; : b = 0, b_{1,1} \neq 0, b_{1,2} = 0, \Delta \neq 0\}$

$U_7 = \{L(\alpha) \in CE(\mu_5) \; : b = 0, b_{1,1} \neq 0, b_{1,2} = \Delta = 0\}$

$U_8 = \{L(\alpha) \in CE(\mu_5) \; : b = b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} \neq 0\}$

$U_9 = \{L(\alpha) \in CE(\mu_5) \; : b = b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} = 0\}$

$U_{10} = \{L(\alpha) \in CE(\mu_5) \; : b = b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} \neq 0\}$

$U_{11} = \{L(\alpha) \in CE(\mu_5) \; : b = b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} = 0\}$

$U_{12} = \{L(\alpha) \in CE(\mu_5) \; : b = b_{1,1} = b_{0,1} = b_{0,0} = 0, b_{1,1} \neq 0\}$

$U_{13} = \{L(\alpha) \in CE(\mu_5) \; : b = b_{1,1} = b_{0,1} = b_{0,0} = b_{1,1} = 0\}$

Proposition 3.6.

   1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if

$$\frac{\Delta \, b^2}{(b_{0,1}b - 2\,b_{1,1})^2} = \frac{\Delta' \, b^2}{(b_{0,1}b' - 2\,b_{1,1})^2}$$

   2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_1$:

$$\frac{\Delta \, b^2}{(b_{0,1}b - 2\,b_{1,1})^2} = \lambda$$

.

   Then algebras from the set U_1 can be parameterized as $L(\lambda, 0, 1, 0, 1), \quad \lambda \in \mathbb{C}$.

Proposition 3.7.

   1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_5 are isomorphic if and only if

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^6 \Delta = \left(\frac{b'_{1,2}}{b'_{1,1}}\right)^6 \Delta'$$

   2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_5$:

$$\left(\frac{b_{1,2}}{4\,b_{1,1}}\right)^6 \Delta = \lambda$$

.

   Then algebras from the set U_5 can be parameterized as $L(\lambda, 0, 1, 1, 0), \quad \lambda \in \mathbb{C}$.

Proposition 3.8.

   1.   *Algebras from U_2 are isomorphic to $L(0,1,0,0,1)$;*

   2.   *Algebras from U_3 are isomorphic to $L(1,0,0,0,1)$;*

   3.   *Algebras from U_4 are isomorphic to $L(0,0,0,0,1)$;*

   4.   *Algebras from U_6 are isomorphic to $L(1,0,1,0,0)$;*

5.   *Algebras from U_7 are isomorphic to $L(0,0,1,0,0)$;*

6.   *Algebras from U_8 are isomorphic to $L(0,1,0,1,0)$;*

7.   *Algebras from U_9 are isomorphic to $L(0,1,0,0,0)$;*

8.   *Algebras from U_{10} are isomorphic to $L(1,0,0,1,0)$;*

9.   *Algebras from U_{11} are isomorphic to $L(1,0,0,0,0)$;*

10.   *Algebras from U_{12} are isomorphic to $L(0,0,0,1,0)$;*

11.   *Algebras from U_{13} are isomorphic to $L(0,0,0,0,0)$.*

3.3   Central extension for 6-dimensional Lie algebra $CE(\mu_6)$

This section is devoted to the classification of $CE(\mu_6)$.
from Lemma (3.1) it is easy to prove $b_{1,4} = -b_{2,3}$

Theorem   3.3.   *(Isomorphism   criterion   for $CE(\mu_6)$))   Two   filiform   Leibniz   algebras α $=$ $(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4})$ and $\alpha' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4})$ from $CE(\mu_6)$ are   isomorphic   iff $\exists A_0, A_1, B_1 \in \mathbb{C}$: such that $A_0 B_1 \neq 0$ and the following equalities hold:*

$$b'_{0,0}   =   \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^5 B_1}, \tag{16}$$

$$b'_{1,1}   =   \frac{B_1 b_{1,1}}{A_0^5}, \tag{17}$$

$$b'_{0,1}   =   \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^5}, \tag{18}$$

$$b'_{1,2}   =   \frac{1}{A_0^4 B_1}(B_1^2 b_{1,2} + (2B_1 B_3 - B_2^2)b_{1,4}), \tag{19}$$

$$b'_{1,4}   =   \frac{B_1}{A_0^2}b_{1,4}. \tag{20}$$

Proof. see prove of theorem **3.1** $\square$

In this section we give a list of all algebras from $CE(\mu_6)$.
Let $\Delta = b_{0,1}^2 - 4\,b_{0,0}\,b_{1,1}$ and $\Delta' = b_{0,1}'^2 - 4\,b_{0,0}'\,b_{1,1}'$ Represent $CE(\mu_6)$ as a union of the following subsets $CE(\mu_6) = \bigcup\limits_{i=1}^{13} U_i$ where

$U_1 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} \neq 0, b_{1,4} \neq 0\}$
$U_2 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} \neq 0, b_{1,4} = 0, b_{1,2} \neq 0\}$
$U_3 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} \neq 0, b_{1,4} = b_{1,2} = 0, \Delta \neq 0\}$
$U_4 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} \neq 0, b_{1,4} = b_{1,2} = \Delta = 0\}$
$U_5 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,4} \neq 0\}$
$U_6 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,4} = 0, b_{1,2} \neq 0\}$
$U_7 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,4} = b_{1,2} = 0\}$
$U_8 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,4} \neq 0\}$
$U_9 = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,4} = 0, b_{1,2} \neq 0\}$
$U_{10} = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,4} = b_{1,2} = 0\}$
$U_{11} = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = b_{0,1} = b_{0,0} = 0, b_{1,4} \neq 0\}$
$U_{12} = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = b_{0,1} = b_{0,0} = b_{1,4} = 0, b_{1,2} \neq 0\}$
$U_{13} = \{L(\alpha) \in CE(\mu_6) : b_{1,1} = b_{0,1} = b_{0,0} = b_{1,4} = b_{1,2} = 0\}$

Proposition 3.9.

    1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if

$$\left(\frac{b_{1,4}}{b_{1,1}}\right)^8 \Delta^3 = \left(\frac{b'_{1,4}}{b'_{1,1}}\right)^8 \Delta'^3$$

    2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_1 :$ $\left(\frac{b_{1,4}}{b_{1,1}}\right)^8 \Delta^3 = \lambda.$

Then algebras from the set U_1 can be parameterized as $L(\lambda, 0, 1, 0, 1),$ $\lambda \in C.$

Proposition 3.10.

    1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_2 are isomorphic if and only if

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^8 \Delta = \left(\frac{b'_{1,2}}{b'_{1,1}}\right)^8 \Delta'$$

    2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_2 :$ $\left(\frac{b_{1,2}}{b_{1,1}}\right)^8 \Delta = \lambda.$

Then algebras from the set U_2 can be parameterized as $L(\lambda, 0, 1, 1, 0),$ $\lambda \in C.$

Proposition 3.11.

    1.   *Algebras from U_3 are isomorphic to $L(1,0,1,0,0)$;*

    2.   *Algebras from U_4 are isomorphic to $L(0,0,1,0,0)$;*

    3.   *Algebras from U_5 are isomorphic to $L(0,1,0,0,1)$;*

    4.   *Algebras from U_6 are isomorphic to $L(0,1,0,1,0)$;*

    5.   *Algebras from U_7 are isomorphic to $L(0,1,0,0,0)$;*

    6.   *Algebras from U_8 are isomorphic to $L(1,0,0,0,1)$;*

    7.   *Algebras from U_9 are isomorphic to $L(1,0,0,1,0)$;*

    8.   *Algebras from U_{10} are isomorphic to $L(1,0,0,0,0)$;*

    9.   *Algebras from U_{11} are isomorphic to $L(0,0,0,0,1)$;*

   10.   *Algebras from U_{12} are isomorphic to $L(0,0,0,1,0)$;*

   11.   *Algebras from U_{13} are isomorphic to $L(0,0,0,0,0)$.*

3.4   Central extension for 7-dimensional Lie algebra $CE(\mu_7)$

From Leibniz Identity we can show that $b_{2,5} = -b_{3,4} = b, b_{2,3} = -b_{1,4}$
Further the elements of $CE(\mu_7)$ will be denoted by $L(\alpha)$ where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b)$ meaning that they are depending on parameters $b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b.$

Theorem 3.4. *(Isomorphism criterion for $CE(\mu_7)$) Two filiform Leibniz algebras $L(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b)$ and $L(b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4}, b')$ from $CE(\mu_7)$ are isomorphic iff $\exists \ A_0, A_1, B_1 \in \mathbb{C} :$ such that $A_0 B_1 (A_0 + A_1 b) \neq 0$,and the following equalities hold:*

$$b'_{0,0} = \frac{A_0{}^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1{}^2 b_{1,1}}{A_0{}^5 B_1 (A_0 + A_1 b)}, \tag{21}$$

$$b'_{0,1} = \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0{}^5 (A_0 + A_1 b)}, \tag{22}$$

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0{}^5 (A_0 + A_1 b)}, \tag{23}$$

$$b'_{1,2} = \frac{B_1{}^2 b_{1,2} + (2 B_1 B_3 - B_2{}^2) b_{1,4} + (2 B_2 B_4 - 2 B_1 B_5 - B_3^3) b}{2 A_0{}^4 B_1 (A_0 + A_1 b)}, \tag{24}$$

$$b'_{1,4} = \frac{-B_1 b_{1,4} + (-2 B_1 B_3 + B_2{}^2) b}{A_0{}^2 B_1 (A_0 + A_1 b)}, \tag{25}$$

$$b' = \frac{B b}{A_0 + A_1 b}. \tag{26}$$

Proof.

Part "if ". Let L_1 and L_2 from $CE(\mu_7)$ be isomorphic: $f : L_1 \cong L_2$. We choose the corresponding adapted basis $\{e_0, e_1, ..., e_7, \}$ in L_1 and $\{e'_0, e'_1.., e'_7\}$ in L_2. Then on this basis the algebras well be presented as $L(\alpha)$ and $L(\alpha')$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In this section we give a list of all algebras from $CE(\mu_7)$.

Represent $CE(\mu_7)$ as a union of the following subsets:

$U_1 = \{L(\alpha) \in CE(\mu_7) : b \neq 0, b_{1,1} \neq 0\}$

$U_2 = \{L(\alpha) \in CE(\mu_7) : b \neq 0, b_{1,1} = 0, b_{0,1} \neq 0\}$

$U_3 = \{L(\alpha) \in CE(\mu_7) : b \neq 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0\}$

$U_4 = \{L(\alpha) \in CE(\mu_7) : b \neq 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\}$

$U_5 = \{L(\alpha) \in CE(\mu_7) : b = 0, b_{1,4} \neq 0, b_{1,1} \neq 0\}$

$U_6 = \{L(\alpha) \in CE(\mu_7) : b = 0, b_{1,4} \neq 0, b_{1,1} = 0, b_{0,1} \neq 0\}$

$U_7 = \{L(\alpha) \in CE(\mu_7) : b = 0, b_{1,4} \neq 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0\}$

$U_8 = \{L(\alpha) \in CE(\mu_7) : b = 0, b_{1,4} \neq 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\}$

$U_9 = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = 0, b_{1,2} \neq 0, b_{1,1} \neq 0\}$

$U_{10} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = 0, b_{1,2} \neq 0, b_{1,1} = 0, b_{0,1} \neq 0\}$

$U_{11} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = 0, b_{1,2} \neq 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0\}$

$U_{12} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = 0, b_{1,2} \neq 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\}$

$U_{13} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = b_{1,2} = 0, b_{1,1} \neq 0, \Delta \neq 0\}$

$U_{14} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = b_{1,2} = 0, b_{1,1} \neq 0, \Delta = 0\}$

$U_{15} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = b_{1,2} = b_{1,1} = 0, b_{0,1} \neq 0\}$

$U_{16} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0\}$

$U_{17} = \{L(\alpha) \in CE(\mu_7) : b = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = 0\}$

Proposition 3.12.

1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if

$$\left(\frac{b}{-2 b_{1,1} + b_{0,1} b}\right)^2 \Delta = \left(\frac{b'}{-2 b'_{1,1} + b'_{0,1} b'}\right)^2 \Delta'$$

2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_1$:

$$\left(\frac{b}{-2 b_{1,1} + b_{0,1} b}\right)^2 \Delta = \lambda$$

.

Then algebras from the set U_1 can be parameterized as $L(\lambda, 0, 1, 0, 0, 1)$, $\lambda \in \mathbb{C}$.

Proposition 3.13.

1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_5 are isomorphic if and only if

$$\left(\frac{b_{1,4}}{b_{1,1}}\right)^{10} \Delta^3 = \left(\frac{b'_{1,4}}{b'_{1,1}}\right)^{10} \Delta'^3$$

2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_5$:

$$\left(\frac{b_{1,4}}{b_{1,1}}\right)^{10} \Delta^3 = \lambda$$

.

Then algebras from the set U_5 can be parameterized as $L(\lambda, 0, 1, 0, 1, 0)$, $\lambda \in \mathbb{C}$.

Proposition 3.14.

1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_9 are isomorphic if and only if

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^{10} \Delta^3 = \left(\frac{b'_{1,2}}{b'_{1,1}}\right)^{10} \Delta'^3$$

2. For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_9$:

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^{10} \Delta^3 = \lambda$$

.

Then algebras from the set U_9 can be parameterized as $L(\lambda, 0, 1, 1, 0, 0)$, $\lambda \in C$.

Proposition 3.15.

1.   *Algebras from U_2 are isomorphic to L(0,1,0,0,0,1);*

2.   *Algebras from U_3 are isomorphic to L(1,0,0,0,0,1);*

3.   *Algebras from U_4 are isomorphic to L(0,0,0,0,0,1);*

4.   *Algebras from U_6 are isomorphic to L(0,1,0,0,1,0);*

5.   *Algebras from U_7 are isomorphic to L(1,0,0,0,1,0);*

6.   *Algebras from U_8 are isomorphic to L(0,0,0,0,1,0);*

7.   *Algebras from U_{10} are isomorphic to L(0,1,0,1,0,0);*

8.   *Algebras from U_{11} are isomorphic to L(1,0,0,1,0,0);*

9.   *Algebras from U_{12} are isomorphic to L(0,0,0,1,0,0);*

10.   *Algebras from U_{13} are isomorphic to L(1,0,1,0,0,0);*

11.   *Algebras from U_{14} are isomorphic to L(0,0,1,0,0,0);*

12.   *Algebras from U_{15} are isomorphic to L(0,1,0,0,0,0);*

13.   *Algebras from U_{16} are isomorphic to L(1,0,0,0,0,0);*

14.   *Algebras from U_{17} are isomorphic to L(0,0,0,0,0,0).*

3.5   Central extension for 8-dimensional Lie algebra $CE(\mu_8)$

It is easy to prove that $b_{1,4} = -b_{3,2}$ and $b_{1,6} = b_{3,4} = b_{5,2}$. The elements of $CE(\mu_8)$ will denoted by $(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, , b_{1,6})$ meaning that they are defined by parameters $b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, , b_{1,6}$

Theorem 3.5. *(Isomorphism criterion for $CE(\mu_8)$) Two filiform Leibniz algebras $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b_{1,6})$ and $\alpha' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4}, b'_{1,6})$ from $CE(\mu_8)$ are isomorphic iff $\exists A_0, A_1, B_i \in \mathbb{C}, \quad 1 \le i \le 5$ such that $A_0 B_1 \ne 0$ and the following equalities hold:*

$$b'_{0,0} \;=\; \frac{A_0{}^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1{}^2 b_{1,1}}{A_0{}^7 B_1}, \tag{27}$$

$$b'_{0,1} \;=\; \frac{2\, A_1 b_{1,1} + A_0 b_{0,1}}{A_0{}^7}, \tag{28}$$

$$b'_{1,1} \;=\; \frac{B_1 b_{1,1}}{A_0{}^7}, \tag{29}$$

$$b'_{1,2} \;=\; \frac{B_1{}^2 b_{1,2} + \left(2\, B_1 B_3 - B_2{}^2\right) b_{1,4} + \left(2\, B_1 B_5 - 2\, B_2 B_4 + B_3{}^2\right) b_{1,6}}{A_0{}^6 B_1}, \tag{30}$$

$$b'_{1,4} \;=\; \frac{B_1{}^2 b_{1,4} + \left(2\, B_1 B_3 - B_2{}^2\right) b_{1,6}}{A_0{}^4 B_1}, \tag{31}$$

$$b'_{1,6} \;=\; \frac{B_1 b_{1,6}}{A_0{}^2}. \tag{32}$$

In this section we give a list of all algebras from $CE(\mu_8)$. Represent $CE(\mu_8)$ as a union of the following subsets :

$U_1 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} \ne 0, b_{1,1} \ne 0\}$
$U_2 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} \ne 0, b_{1,1} = 0, b_{0,1} \ne 0\}$
$U_3 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} \ne 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \ne 0\}$
$U_4 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} \ne 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\}$
$U_5 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = 0, b_{1,4} \ne 0, b_{1,1} \ne 0\}$
$U_6 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = 0, b_{1,4} \ne 0, b_{1,1} = 0, b_{0,1} \ne 0\}$
$U_7 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = 0, b_{1,4} \ne 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \ne 0\}$
$U_8 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = 0, b_{1,4} \ne 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\}$
$U_9 = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = 0, b_{1,2} \ne 0, b_{1,1} \ne 0\}$
$U_{10} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = 0, b_{1,2} \ne 0, b_{1,1} = 0, b_{0,1} \ne 0\}$
$U_{11} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = 0, b_{1,2} \ne 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \ne 0\}$
$U_{12} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = 0, b_{1,2} \ne 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\}$
$U_{13} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = b_{1,2} = 0, b_{1,1} \ne 0, \Delta \ne 0\}$
$U_{14} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = b_{1,2} = 0, b_{1,1} \ne 0, \Delta = 0\}$
$U_{15} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = b_{1,2} = b_{1,1} = 0, b_{0,1} \ne 0\}$
$U_{16} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = 0, b_{0,0} \ne 0\}$
$U_{17} = \{L(\alpha) \in CE(\mu_8) \; : b_{1,6} = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = 0\}$ In U_1 the following proposition is holds

Proposition 3.16.

1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if

$$\left(\frac{b_{1,6}}{b_{1,1}}\right)^{12} \Delta^5 = \left(\frac{b'_{1,6}}{b'_{1,1}}\right)^{12} \Delta'^5$$

2. For any λ from $\mathbb{C}$ there exists $L(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b_{1,6}) \in U_1$:

$$\left(\frac{b_{1,6}}{b_{1,1}}\right)^{12} \Delta^5 = \lambda$$

.

Then algebras from the set U_1 can be parameterized as $L(\lambda, 0, 1, 0, 0, 1)$, $\lambda \in \mathbb{C}$.

In U_2 the following proposition is holds

Proposition 3.17.

1. Two algebras $L(\alpha)$ and $L(\alpha')$ from U_5 are isomorphic if and only if

$$\left(\frac{b_{1,1}}{b_{1,4}}\right)^4 \frac{1}{\Delta} = \left(\frac{b'_{1,1}}{b'_{1,4}}\right)^4 \frac{1}{\Delta'}$$

2. For any λ from $\mathbb{C}$ there exists $L(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b_{1,6}) \in U_5$:

$$\left(\frac{b_{1,1}}{b_{1,4}}\right)^4 \frac{1}{\Delta} = \lambda$$

.

Then algebras from the set U_5 can be parameterized as $L(\lambda, 0, 1, 0, 1, 0)$, $\lambda \in \mathbb{C}$.

In U_9 the following proposition is holds

Proposition 3.18.

1. Two algebras $L(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b_{1,6})$ and $L(b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4}, b'_{1,6})$ from U_9 are isomorphic if and only if

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^{12} \Delta = \left(\frac{b'_{1,2}}{b'_{1,1}}\right)^{12} \Delta'$$

2. For any λ from $\mathbb{C}$ there exists $L(b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b_{1,6}) \in U_3$:

$$\left(\frac{b_{1,2}}{b_{1,1}}\right)^{12} \Delta = \lambda$$

.

Then algebras from the set U_9 can be parameterized as $L(\lambda, 0, 1, 1, 0, 0)$, $\lambda \in \mathbb{C}$.

Proposition 3.19.

1.   *Algebras from U_2 are isomorphic to $L(0,1,0,0,0,1)$;*

2.   *Algebras from U_3 are isomorphic to $L(1,0,0,0,0,1)$;*

3.   *Algebras from U_4 are isomorphic to $L(0,0,0,0,0,1)$;*

4.   *Algebras from U_6 are isomorphic to $L(0,1,0,0,1,0)$;*

5.   *Algebras from U_7 are isomorphic to $L(1,0,0,0,1,0)$;*

6.   *Algebras from U_8 are isomorphic to $L(0,0,0,0,1,0)$;*

7.   *Algebras from U_{10} are isomorphic to $L(0,1,0,1,0,0)$;*

8.   *Algebras from U_{11} are isomorphic to $L(1,0,0,1,0,0)$;*

9.   *Algebras from U_{12} are isomorphic to $L(0,0,0,1,0,0)$;*

10.   *Algebras from U_{13} are isomorphic to $L(1,0,1,0,0,0)$;*

11.   *Algebras from U_{14} are isomorphic to $L(0,0,1,0,0,0)$;*

12.   *Algebras from U_{15} are isomorphic to $L(0,1,0,0,0,0)$;*

13.   *Algebras from U_{16} are isomorphic to $L(1,0,0,0,0,0)$;*

14.   *Algebras from U_{17} are isomorphic to $L(0,0,0,0,0,0)$.*

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